JET SPACES IN COMPLEX ANALYTIC GEOMETRY: AN EXPOSITION

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In [7] Pillay described the model-theoretic significance of a result in complex analytic geometry due to Campana [1] and Fujiki [2]. These notes are an exposition of the "jet space" constructions that underly the Campana/Fujiki theorem. In particular, we discuss infinitesimal neighbourhoods as well as the sheaves of principal parts, jets, and differential operators. The material is drawn largely from Grothendieck [4] and Kantor [5]. We also describe how these constructions are used by Campana and Fujiki.

1. Some sheaf-theoretic preliminaries

Suppose $f : X \to Y$ is a continuous map of topological spaces, $\mathcal{F}$ is a sheaf on X, and $\mathcal{G}$ is a sheaf on Y.

- The *direct image of* $\mathcal{F}$, denoted by $f_*\mathcal{F}$, is the sheaf on Y which assigns to every open set $V \subseteq Y$ the object $\mathcal{F}(f^{-1}(V))$.
- The *inverse image of* $\mathcal{G}$, denoted by $f^{-1}\mathcal{G}$, is the sheaf on X associated to the pre-sheaf which assigns to every open set $U \subseteq X$ the object

$$\varinjlim \{\mathcal{G}(V) : V \supseteq f(U)\}.$$

Note that if $X = \{y\}$ is a closed point of Y, and f is the inclusion map, then $f^{-1}\mathcal{G} = \mathcal{G}_y$, the stalk of $\mathcal{G}$ at y.

The functors f^{-1} and f_* have an adjointness property: there are natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$, so that $\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \approx \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$.

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, then a morphism is given by a continuous map on topological spaces, $f : X \to Y$, together with a map of sheaves on Y, $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. By adjointness, a morphism can also be given by a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X.

If $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then $f^{-1}\mathcal{G}$ is a sheaf of $f^{-1}\mathcal{O}_Y$-modules. On the other hand, $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ makes $\mathcal{O}_X$ into a sheaf of $f^{-1}\mathcal{O}_Y$-algebras. Hence we can make $f^{-1}\mathcal{G}$ into a sheaf of $\mathcal{O}_X$-modules by tensoring with $\mathcal{O}_X$ over $f^{-1}\mathcal{O}_Y$. This is denoted by $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, and is sometimes referred to as the *$\mathcal{O}_X$-module inverse image of $\mathcal{G}$*.

In the particular case when $X = \{y\}$ is a closed (reduced) point of a complex analytic space Y, and f is the inclusion map, then $f^*\mathcal{G}$ is the complex vector space $\mathcal{G}_y \otimes_{\mathcal{O}_{Y,y}} \mathbb{C}$.

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Remark 1.1. Let $f : X \to Y$ be a morphism of ringed spaces. Suppose $Y' \subseteq Y$ is a closed subspace whose defining ideal sheaf is $\mathcal{J}$. The inclusion map $\mathcal{J} \to \mathcal{O}_Y$ induces an inclusion $f^{-1}\mathcal{J} \to f^{-1}\mathcal{O}_Y$, and hence a canonical $\mathcal{O}_X$-linear map

$$f^*\mathcal{J} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = \mathcal{O}_X.$$

Note, however, that this map may not by an inclusion, and so $f^*\mathcal{J}$ need not be an ideal sheaf of $\mathcal{O}_X$. We therefore define the *inverse image ideal sheaf of* $\mathcal{J}$, denoted by $f^{-1}\mathcal{J} \cdot \mathcal{O}_X$, to be the image of $f^*\mathcal{J}$ in $\mathcal{O}_X$ under the above map. This notation is justified by the fact that if $x \in X$ and $f(x) = y \in Y'$, then $(f^{-1}\mathcal{J} \cdot \mathcal{O}_X)_x$ is in fact the ideal sheaf generated by the image of $f^{-1}(\mathcal{J}_y)$ under the inclusion $f^{-1}\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Note also that $\mathcal{O}_X/(f^{-1}\mathcal{J} \cdot \mathcal{O}_X) \approx f^*(\mathcal{O}_Y/\mathcal{J})$. Finally, we say that a closed subspace $X' \subseteq X$ is the *inverse image of* Y' if on underlying spaces $f^{-1}(Y') = X'$ and the defining ideal sheaf of X' is $f^{-1}\mathcal{J} \cdot \mathcal{O}_X$.

Lemma 1.2. *Suppose we have the following commuting diagram of morphisms of ringed spaces*

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "g"'] \arrow[dr, "p"] & X \arrow[d, "f"] \\ Y' \arrow[r, "j"] & Y \end{tikzcd}$$

where i and j are closed immersions. Then $i(X')$ is the inverse image of $j(Y')$ if and only if

$$g^{-1}\mathcal{O}_{Y'} \otimes_{p^{-1}\mathcal{O}_Y} i^{-1}\mathcal{O}_X \;\approx\; \mathcal{O}_{X'}$$

under the canonical map.

Proof. Let $\mathcal{I}$ be the defining ideal sheaf of $i(X')$ and $\mathcal{J}$ the defining ideal sheaf of $j(Y')$. Then, as $\mathcal{O}_X/(f^{-1}\mathcal{J} \cdot \mathcal{O}_X) = f^*(\mathcal{O}_Y/\mathcal{J})$ we have that $\mathcal{I} = f^{-1}\mathcal{J} \cdot \mathcal{O}_X$ if and only if

$$\mathcal{O}_X/\mathcal{I} \;=\; f^{-1}(\mathcal{O}_Y/\mathcal{J}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Now, applying i^{-1}, this is equivalent to:

$$i^{-1}(\mathcal{O}_X/\mathcal{I}) \;=\; p^{-1}(\mathcal{O}_Y/\mathcal{J}) \otimes_{p^{-1}\mathcal{O}_Y} i^{-1}\mathcal{O}_X$$

Observing that $i^{-1}(\mathcal{O}_X/\mathcal{I}) = \mathcal{O}_{X'}$ and $j^{-1}(\mathcal{O}_Y/\mathcal{J}) = \mathcal{O}_{Y'}$, this is in turn equivalent to $\mathcal{O}_{X'} = g^{-1}\mathcal{O}_{Y'} \otimes_{p^{-1}\mathcal{O}_Y} i^{-1}\mathcal{O}_X$, as desired. □

2. Infinitesimal neighbourhoods

The construction we describe here can be found in Grothendieck, Section 1 of [4].

Let $i : Y \to X$ be a closed immersion of complex analytic spaces. It follows that the induced map of sheaves on Y, $i^{-1}\mathcal{O}_X \to \mathcal{O}_Y$, is a surjection. Let $\mathcal{I} := \ker(i^{-1}\mathcal{O}_X \to \mathcal{O}_Y)$. For each $n \geq 0$, the *nth infinitesimal neighbourhood of Y in X* is the complex analytic space

$$Y_i^{(n)} := (Y, i^{-1}\mathcal{O}_X/\mathcal{I}^{n+1}).$$

Remark 2.1. If $Y \subseteq X$ is a closed analytic subspace defined by the ideal sheaf $\mathcal{J}$ of $\mathcal{O}_X$, and i is the inclusion map, then we have a natural identification of $Y_i^{(n)}$ with $(Y, \mathcal{O}_X/\mathcal{J}^{n+1})$. Indeed, note that we view $\mathcal{O}_X/\mathcal{J}^{n+1}$ as a sheaf on Y by restricting this sheaf on X to its support. In other words, as a sheaf on Y,

$\mathcal{O}_X/\mathcal{J}^{n+1} = i^{-1}(\mathcal{O}_X/\mathcal{J}^{n+1})$. We wish to observe that $\mathcal{O}_{Y_i^{(n)}}$ is isomorphic to $i^{-1}(\mathcal{O}_X/\mathcal{J}^{n+1})$. By the exactness of i^{-1}, we obtain

$$0 \longrightarrow i^{-1}(\mathcal{J}^{n+1}) \longrightarrow i^{-1}\mathcal{O}_X \longrightarrow i^{-1}(\mathcal{O}_X/\mathcal{J}^{n+1}) \longrightarrow 0$$

It suffices therefore to observe that $\mathcal{I}^{n+1} = i^{-1}(\mathcal{J}^{n+1})$. As i^{-1} commutes with products of ideal sheaves, we need $\mathcal{I} = i^{-1}\mathcal{J}$. But this follows from the above exact sequence for $n = 0$, together with the fact that $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J} = i^{-1}(\mathcal{O}_X/\mathcal{J})$.

There is a canonical closed immersion $i^{(n)} : Y_i^{(n)} \to X$. This map is given by i on the underlying spaces and by the quotient map $i^{-1}\mathcal{O}_X \to \mathcal{O}_{Y_i^{(n)}} = i^{-1}\mathcal{O}_X/\mathcal{I}^{n+1}$ on the structure sheaves.

There is also the closed immersion $i^{(n,0)} : Y \to Y_i^{(n)}$. It is the identity on underlying spaces and the quotient map $i^{-1}\mathcal{O}_X/\mathcal{I}^{n+1} \to \mathcal{O}_Y$ on structure sheaves. More generally, for $m \geq n$, there is $i^{(m,n)} : Y_i^{(n)} \to Y_i^{(m)}$ given by the identity on underlying spaces and the quotient maps $i^{-1}\mathcal{O}_X/\mathcal{I}^{m+1} \to i^{-1}\mathcal{O}_X/\mathcal{I}^{n+1}$ on sheaves. Note that the following commute:

$$\begin{tikzcd}
Y_i^{(n)} \arrow[rr, "i^{(m,n)}"] \arrow[dr, "i^{(n)}"'] & & Y_i^{(m)} \arrow[dl, "i^{(m)}"] \\
& X &
\end{tikzcd}$$

The construction of the infinitesimal neighbourhood is functorial. Given

$$\begin{tikzcd}
Y' \arrow[r, "i'"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

where i and i' are closed immersions, there is a unique morphism $\alpha : Y'^{(n)}_{i'} \to Y_i^{(n)}$ such that the following commutes:

$$\begin{tikzcd}
Y' \arrow[r, "i^{(0,n)}"] \arrow[d, "f"'] & Y'^{(n)}_{i'} \arrow[r, "i'^{(n)}"] \arrow[d, "\alpha"] & X' \arrow[d, "g"] \\
Y \arrow[r, "i^{(0,n)}"] & Y_i^{(n)} \arrow[r, "i^{(n)}"] & X
\end{tikzcd}$$

Indeed, considering an isomorphic copy of the situation, we may assume $Y \subseteq X$ with defining ideal sheaf $\mathcal{J}$, $Y' \subseteq X'$ with defining ideal sheaf $\mathcal{J}'$, both i and i' are inclusions, and f is the restriction of g to Y'. As in Remark 2.1, $\mathcal{O}_{Y_i^{(n)}} = \mathcal{O}_X/\mathcal{J}^{n+1}$ and $\mathcal{O}_{Y'^{(n)}_{i'}} = \mathcal{O}_{X'}/\mathcal{J}'^{n+1}$. The map $g^{-1}\mathcal{O}_X \to \mathcal{O}_{X'}$, given by g on structure sheaves, induces

$$f^{-1}\mathcal{O}_Y = g^{-1}(\mathcal{O}_X/\mathcal{J}) = g^{-1}\mathcal{O}_X/g^{-1}\mathcal{J} \longrightarrow \mathcal{O}_{X'}/\mathcal{J}' = \mathcal{O}_{Y'}.$$

It therefore takes $(g^{-1}\mathcal{J})^{n+1}$ to $\mathcal{J}'^{n+1}$ and hence induces

$$f^{-1}\mathcal{O}_{Y_i^{(n)}} = g^{-1}(\mathcal{O}_X/\mathcal{J}^{n+1}) = g^{-1}\mathcal{O}_X/(g^{-1}\mathcal{J})^{n+1} \longrightarrow \mathcal{O}_{X'}/\mathcal{J}'^{n+1} = \mathcal{O}_{Y'^{(n)}_{i'}}.$$

This map, together with f on the underlying spaces, gives $\alpha : Y'^{(n)}_{i'} \to Y_i^{(n)}$. Uniqueness follows immediately from the commuting requirement.

Lemma 2.2. *Suppose $Y \subseteq X$ and $Y' \subseteq X'$ are closed subspaces of complex analytic spaces, $i : Y \to X$ and $i' : Y' \to X'$ are the inclusions, and Y' is the inverse image of Y under a morphism $g : X' \to X$. Then $Y'^{(n)}_{i'}$ is the inverse image of $Y^{(n)}_i$.*

Proof. Let $\mathcal{I}$ be the defining ideal of Y in X and $\mathcal{J}$ the defining ideal of Y' in X'. That Y' is the inverse image of Y under g means that $g^{-1}\mathcal{I} \cdot \mathcal{O}_{X'} = \mathcal{J}$ (see Remark 1.1). But as $(g^{-1}\mathcal{I} \cdot \mathcal{O}_{X'})^{n+1} = g^{-1}(\mathcal{I}^{n+1}) \cdot \mathcal{O}_{X'}$, we see that $\mathcal{J}^{n+1}$ is the inverse image ideal sheaf of $\mathcal{I}^{n+1}$. Now observe that $\mathcal{I}^{n+1}$ is the defining ideal of $Y^{(n)}_i$ in X and $\mathcal{J}^{n+1}$ is the defining ideal of $Y'^{(n+1)}_{i'}$ in X'. $\square$

3. Relative principal parts

The following construction can be found in Grothendieck, Section 2 of [4].

Let $p : X \to Y$ be a morphism of complex analytic spaces. The diagonal map $\mathrm{diag} : X \to X \times_Y X$ is a closed immersion into the fibred product.[1] Let

$$\Delta^{(n)}_{X/Y} := X^{(n)}_{\mathrm{diag}}$$

be the nth infinitesimal neighbourhood of X in $X \times_Y X$ via the diagonal map. That is, if $\mathcal{I}$ denotes the kernel of the induced surjection $\mathrm{diag}^{-1}\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X$, then $\Delta^{(n)}_{X/Y} = (X, \mathrm{diag}^{-1}\mathcal{O}_{X\times_Y X}/\mathcal{I}^{n+1})$.

Letting $D \subseteq X \times_Y X$ be the diagonal, as a closed analytic subspace of $X \times_Y X$ over Y, and $\mathcal{J}$ the defining ideal sheaf of D, we see by Remark 2.1 that $\Delta^{(n)}_{X/Y}$ is isomorphic to $(D, \mathcal{O}_{X\times_Y X}/\mathcal{J}^{n+1})$. This latter is the nth infinitesimal neighbourhood of the diagonal in $X \times_Y X$, and is what Fujiki uses in [2].

We wish to view $\mathcal{O}_{\Delta^{(n)}_{X/Y}}$ as a sheaf of $\mathcal{O}_X$-algebras in a natural way. This is done by viewing $\Delta^{(n)}_{X/Y}$ as an analytic space over X via the following morphism:

$$\Delta^{(n)}_{X/Y} \xrightarrow{\mathrm{diag}^{(n)}} X \times_Y X \xrightarrow{\mathrm{pr}_1} X$$

Indeed, as $\alpha := \mathrm{pr}_1 \circ \mathrm{diag}^{(n)}$ is the identity on the underlying spaces, $\alpha^{-1}\mathcal{O}_X = \mathcal{O}_X$ and $\alpha_*\mathcal{O}_{\Delta^{(n)}_{X/Y}} = \mathcal{O}_{\Delta^{(n)}_{X/Y}}$. We thus obtain a map of sheaves $\mathcal{O}_X \to \mathcal{O}_{\Delta^{(n)}_{X/Y}}$, which we denote[2] by pr_1^*. Note that at stalks this map is just

$$\mathcal{O}_{X,x} \xrightarrow{\mathrm{pr}_1^\#} \mathcal{O}_{X\times_Y X,(x,x)} \to \mathcal{O}_{X\times_Y X,(x,x)}/\mathcal{J}^{n+1}_{(x,x)} = \mathcal{O}_{\Delta^{(n)}_{X/Y},x}$$

where $\mathcal{J}$ is the ideal sheaf of the diagonal in $X \times_Y X$.

By the *principal parts of order n of X relative to Y*, denoted by $\mathcal{P}^{(n)}_{X/Y}$, we mean the structure sheaf of $\Delta^{(n)}_{X/Y}$ viewed as a sheaf of $\mathcal{O}_X$-algebras via the map $\mathrm{pr}_1^* : \mathcal{O}_X \to \mathcal{O}_{\Delta^{(n)}_{X/Y}}$. That is, $\mathcal{P}^{(n)}_{X/Y} = \mathrm{diag}^{-1}\mathcal{O}_{X\times_Y X}/\mathcal{I}^{n+1}$ together with its sheaf of $\mathcal{O}_X$-algebras structure described above.

[1] We only consider separated topological spaces.

[2] This notation is used by Grothendieck [4], and is short for $(\mathrm{pr}_1 \circ \mathrm{diag}^{(n)})^\#$.

Remark 3.1. There is also the morphism $\mathrm{pr}_2 \circ \mathrm{diag}^{(n)} : \Delta^{(n)}_{X/Y} \to X$, which (being the identity on underlying spaces) produces the map of sheaves on X,

$$\mathrm{d}^{(n)}_{X/Y} := \mathrm{pr}_2^* : \mathcal{O}_X \to \mathcal{P}^{(n)}_{X/Y}$$

which is called the *universal differential operator of order n for X relative to Y*. Note that this map is *not* $\mathcal{O}_X$-linear. For example, suppose Y is a point and $X = \mathbb{C}$. By Remark 2.1, $\Delta^{(1)}_{\mathbb{C}} \approx (D, \mathcal{O}_{\mathbb{C}^2}/\mathcal{J})$, where $\mathcal{O}_{\mathbb{C}^2}$ is the sheaf of germs of holomorphic functions on the complex plane, $D \subseteq \mathbb{C}^2$ is the diagonal, and $\mathcal{J}$ is its defining ideal sheaf. Taking stalks at the origin, $\mathcal{O}_{\Delta^{(1)}_{\mathbb{C}},0} \approx \mathbb{C}[[x,y]]/(x-y)^2$, where $\mathbb{C}[[x,y]]$ denotes the ring of convergent power series at the origin in $\mathbb{C}^2$. It is then not hard to see that $\mathcal{P}^{(1)}_{\mathbb{C},0}$ is isomorphic to $\mathbb{C}[[x,y]]/(x-y)^2$ together with the $\mathbb{C}[[u]]$-algebra structure given by $u \mapsto x \operatorname{mod}(x-y)^2$. On the other hand $\mathrm{pr}_2^* : \mathcal{O}_{\mathbb{C},0} \to \mathcal{P}^{(1)}_{\mathbb{C},0}$ produces the map $\mathbb{C}[[u]] \to \mathbb{C}[[x,y]]/(x-y)^2$ given by $u \mapsto y \operatorname{mod}(x-y)^2$. This is not $\mathbb{C}[[u]]$-linear, since for example

$$\mathrm{pr}_2^*(u \cdot u) = y^2 \operatorname{mod}(x-y)^2 \neq xy \operatorname{mod}(x-y)^2 = u \cdot \mathrm{pr}_2^*(u).$$

Lemma 3.2. *As sheaves of $\mathcal{O}_X$-modules, $\mathcal{P}^{(n)}_{X/Y} \approx \mathcal{O}_X \oplus \mathcal{I}/\mathcal{I}^{n+1}$.*

Proof. Some words of explanation. Note that each of $\mathcal{P}^{(n)}_{X/Y}$, $\mathcal{O}_X$, and $\mathcal{I}/\mathcal{I}^{n+1}$ is equipped with a natural $\mathcal{O}_X$-module structure. Indeed, on $\mathcal{P}^{(n)}_{X/Y}$ it is given by what we have been writing as $\mathrm{pr}_1^* : \mathcal{O}_X \to \mathcal{P}^{(n)}_{X/Y}$, and on $\mathcal{I}/\mathcal{I}^{n+1}$ the $\mathcal{O}_X$-module structure is inherited from being an ideal sheaf of $\mathcal{P}^{(n)}_{X/Y}$.

In order to prove the Lemma, it suffice to show that $\mathrm{pr}_1^* : \mathcal{O}_X \to \mathcal{P}^{(n)}_{X/Y}$ is a section to the quotient map $\mathcal{P}^{(n)}_{X/Y} \longrightarrow \mathcal{O}_X \approx \mathrm{diag}^{-1}(\mathcal{O}_{X\times_Y X})/\mathcal{I}$. Indeed, this would imply that the following exact sequence of $\mathcal{P}^{(n)}_{X/Y}$-modules

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^{n+1} \longrightarrow \mathcal{P}^{(n)}_{X/Y} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

is in fact a split exact sequence of $\mathcal{O}_X$-modules, hence $\mathcal{P}^{(n)}_{X/Y} \approx \mathcal{O}_X \oplus \mathcal{I}/\mathcal{I}^{n+1}$ as desired.

Recall that pr_1^* is the map on structure sheaves coming from the morphism $\mathrm{pr}_1 \circ \mathrm{diag}^{(n)} : \Delta^{(n)}_{X/Y} \to X$. On the other hand, the quotient map $\mathcal{P}^{(n)}_{X/Y} \longrightarrow \mathcal{O}_X$ is the map on structure sheaves determined by the morphism $\mathrm{diag}^{(n,0)} : X \to \Delta^{(n)}_{X/Y}$. That $\mathrm{pr}_1^* : \mathcal{O}_X \to \mathcal{P}^{(n)}_{X/Y}$ is a section to $\mathcal{P}^{(n)}_{X/Y} \longrightarrow \mathcal{O}_X$ then follows from the fact that diag is a section pr_1 and from following commuting diagram (Section 2):

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{diag}^{(n,0)}"] \arrow[dr, "\mathrm{diag}"'] & & \Delta^{(n)}_{X/Y} \arrow[dl, "\mathrm{diag}^{(n)}"] \\
& X \times_Y X &
\end{tikzcd}$$

□

The isomorphism of Lemma 3.2 can also be viewed as one of $\mathcal{O}_X$-algebras, where $\mathcal{O}_X \oplus \mathcal{I}/\mathcal{I}^{n+1}$ has the natural algebra structure in which $\mathcal{I}/\mathcal{I}^{n+1}$ is a nilpotent

ideal sheaf of order $n+1$. This algebra structure is given by: multiplication on $\mathcal{O}_X$ as a sheaf of rings, multiplication on $\mathcal{I}/\mathcal{I}^{n+1}$ as an ideal sheaf of $\mathcal{P}^{(n)}_{X/Y}$, and multiplication between the two by the $\mathcal{O}_X$-module structure on $\mathcal{I}/\mathcal{I}^{n+1}$.

The sheaf of $\mathcal{O}_X$-modules $\mathcal{I}/\mathcal{I}^{n+1}$ is denoted by $\underline{\Omega}^n_{X/Y}$ and called the *sheaf of relative n-differentials of X over Y.*

Note that $\mathcal{P}^{(n)}_{X/Y}$ is a coherent sheaf of $\mathcal{O}_X$-algebras of finite type. Indeed, this follows from the fact that both $\mathcal{O}_X$ and $\underline{\Omega}^n_{X/Y}$ are coherent sheaves of $\mathcal{O}_X$-modules of finite type.

We conclude with the following concrete description of the stalks of $\mathcal{P}^{(n)}_{X/Y}$.

Proposition 3.3. *Suppose $x \in X$ and $p(x) = y \in Y$. There is a canonical isomorphism of $\mathbb{C}$-vector spaces,*

$$\mathcal{P}^{(n)}_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C} \xrightarrow{\approx} \mathcal{O}_{X_y,x}/\mathcal{M}^{n+1}_{X_y,x}$$

where X_y is the fibre of $p : X \to Y$ above y, and $\mathcal{M}_{X_y,x}$ is its maximal ideal at x. In particular, when Y is a point we have

$$\mathcal{P}^{(n)}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C} \approx \mathcal{O}_{X,x}/\mathcal{M}^{n+1}_{X,x}$$

Proof. This is proved in [4] in full generality. We consider only the case when Y is a point. Let $i : Y \to X$ be the section to $p : X \to Y$, given by $i(Y) = x$. Consider the following commuting square:

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\mathrm{diag}"] \\ X \arrow[r, "(ip{,}\mathrm{id})"'] & X \times X \end{tikzcd}$$

Under the vertical arrows, we can identify Y and X with closed subspaces of X and $X \times X$ respectively. As such, Y is the inverse image of X under (ip, id) (see Remark 1.1). Taking infinitesimal neighbourhoods with respect to the vertical arrows we obtain (by functoriality)

$$\begin{tikzcd} Y_i^{(n)} \arrow[r, "\alpha"] \arrow[d, "i^{(n)}"'] & \Delta_X^{(n)} \arrow[d, "\mathrm{diag}^{(n)}"] \\ X \arrow[r, "(ip{,}\mathrm{id})"'] & X \times X \end{tikzcd}$$

Moreover, it follows by Lemma 2.2, that $Y_i^{(n)}$ is the inverse image of $\Delta_X^{(n)}$ under (ip, id). It follows that $Y_i^{(n)}$ is also the inverse image of X under $\mathrm{diag}^{(n)}$ – where we use the horizontal arrows (which are also closed immersions) to identify $Y_i^{(n)}$ and X with closed subspaces of $\Delta_X^{(n)}$ and $X \times X$ respectively. Indeed, the characterisation given by Lemma 1.2 shows that being the inverse image is independent of the orientation we consider.

On the other hand, considering

$$\begin{tikzcd} X \arrow[r, "(ip,\mathrm{id})"] \arrow[d, "p"'] & X \times X \arrow[d, "\mathrm{pr}_1"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

we see that X is itself the inverse image of Y under pr_1 – using the horizontal arrows for the appropriate identifications. Putting these together yields

$$\begin{tikzcd} Y_i^{(n)} \arrow[r, "\alpha"] \arrow[d, "p\circ i^{(n)}"'] & \Delta_X^{(n)} \arrow[d, "\mathrm{pr}_1 \circ \mathrm{diag}^{(n)}"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

and $Y_i^{(n)}$ is the inverse image of Y under $\mathrm{pr}_1 \circ \mathrm{diag}^{(n)}$ – where we use the horizontal arrows to identify $Y_i^{(n)}$ and Y with closed subspaces of $\Delta_X^{(n)}$ and X respectively.

By Lemma 1.2, and noting that $\mathrm{pr}_1 \circ \mathrm{diag}^{(n)}$ is what gives the structure sheaf of $\Delta_X^{(n)}$ the $\mathcal{O}_X$-module structure of $\mathcal{P}_X^{(n)}$, we get that

$$(p \circ i^{(n)})^{-1}\mathcal{O}_Y \otimes_{(i\circ p\circ i^{(n)})^{-1}\mathcal{O}_X} \alpha^{-1}\mathcal{P}_X^{(n)} \quad \approx \quad \mathcal{O}_{Y_i^{(n)}}$$

As $p \circ i^{(n)}$ is the identity on underlying spaces and α and i agree on underlying spaces, this yields

$$\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{P}_X^{(n)} \quad \approx \quad \mathcal{O}_{Y_i^{(n)}}$$

But as Y is a point and $i : Y \to X$ is the section $i(Y) = x$, we have the following degenerations: $i^{-1}\mathcal{P}_X^{(n)} = \mathcal{P}_{X,x}^{(n)}$, $i^{-1}\mathcal{O}_X = \mathcal{O}_{X,x}$, $\mathcal{O}_Y = \mathbb{C}$, and

$$\mathcal{O}_{Y_i^{(n)}} = i^{-1}\mathcal{O}_X/[\ker(i^{-1}\mathcal{O}_X \to \mathcal{O}_Y)]^{n+1} = \mathcal{O}_{X,x}/\mathcal{M}_{X,x}^{n+1}$$

Plugging these into the isomorphism displayed above proves the Proposition. □

Remark 3.4. Let $\Delta_X^{(n)} \to X$ be the map $\mathrm{pr}_1 \circ \mathrm{diag}^{(n)}$. Then Proposition 3.3 tells us that for $a \in X$, the (sheaf-theoretic) fibre of $\Delta_X^{(n)}$ over a is (canonically isomorphic to) the analytic space $(\{a\}, \mathcal{O}_{X,a}/\mathcal{M}_{X,a}^{n+1})$.

Remark 3.5. Suppose $i : Y \to X$ is a section to $p : X \to Y$ that is also a closed immersion. The structure sheaf of the nth infinitesimal neighbourhood of Y in X, $\mathcal{O}_{Y_i^{(n)}}$, is given the structure of an $\mathcal{O}_Y$-algebra by $p \circ i^{(n)} : Y_i^{(n)} \to Y$. The argument for Proposition 3.3 given above actually proves: *There is a canonical $\mathcal{O}_Y$-linear isomorphism $i^*\mathcal{P}_{X/Y}^{(n)} \xrightarrow{\approx} \mathcal{O}_{Y_i^{(n)}}$.*

Corollary 3.6. *Suppose $x \in X$ and $p(x) = y \in Y$. There is a canonical isomorphism of $\mathbb{C}$-vector spaces,*

$$\underline{\Omega}_{X/Y,x}^{(n)} \otimes_{\mathcal{O}_{X,x}} \mathbb{C} \xrightarrow{\approx} \mathcal{M}_{X_y,x}/\mathcal{M}_{X_y,x}^{n+1}.$$

In particular, when Y is a point we have

$$\underline{\Omega}_{X,x}^{(n)} \otimes_{\mathcal{O}_{X,x}} \mathbb{C} \approx \mathcal{M}_{X,x}/\mathcal{M}_{X,x}^{n+1}$$

Proof. This is an immediate consequence of Lemma 3.2 and Proposition 3.3. □

4. Jets

The material in this section is drawn from Kantor [5].

Let X be a complex analytic space, and $\pi : \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X$ the surjective map of sheaves on X determined by the presheaf map $\mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ given by $f \otimes g \mapsto fg$. Let $\mathcal{J} := \ker \pi$, and define the *sheaf of jets on* X to be

$$Jet_X^{(n)} := (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X)/\mathcal{J}^{n+1}.$$

We view $Jet_X^{(n)}$ as a sheaf of $\mathcal{O}_X$-algebras by the composition

$$\mathcal{O}_X \xrightarrow{p_1} \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X)/\mathcal{J}^{n+1} = Jet_X^{(n)}$$

where the second arrow is the quotient map and the first is determined by the presheaf map $\mathcal{O}_X(U) \to \mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U)$ given by $f \mapsto f \otimes 1$.

Remark 4.1. We let $d_X^{(n)} : \mathcal{O}_X \to Jet_X^{(n)}$ be the composition of $p_2 : \mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X$ with the quotient map, where p_2 is determined by $f \mapsto 1 \otimes f$. This should not be confused with $\mathrm{d}_X^{(n)} : \mathcal{O}_X \to \mathcal{P}_X^{(n)}$, the "universal differential operator" discussed in Remark 3.1.

Proposition 4.2. *Suppose $(X, \mathcal{O}_X)$ is a reduced complex analytic space. Then there is a canonical $\mathcal{O}_X$-linear map, $Jet_X^{(n)} \to \mathcal{P}_X^{(n)}$, such that*

$$\begin{tikzcd} Jet_X^{(n)} \arrow[rr] & & \mathcal{P}_X^{(n)} \\ & \mathcal{O}_X \arrow[ul, "d_X^{(n)}"] \arrow[ur, "\mathrm{d}_X^{(n)}"'] & \end{tikzcd}$$

commutes. Moreover, if X is smooth, then $Jet_X^{(n)} \to \mathcal{P}_X^{(n)}$ is an isomorphism.

Proof. Fix an open set $U \subseteq X$. As X is reduced we may interpret the elements of $\mathcal{O}_X(U)$ as (holomorphic) $\mathbb{C}$-valued functions on U. Consider the map

$$\alpha_U : \mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \to \mathcal{O}_{X\times X}(U \times U)$$

given by $f \otimes g \mapsto f(x)g(y)$. On the other hand, we have the natural map

$$\beta_U : \mathcal{O}_{X\times X}(U \times U) \longrightarrow \lim_{V \supseteq \mathrm{diag}(U)} \mathcal{O}_{X\times X}(V)$$

The composition, $\beta_U \circ \alpha_U$, determines a map, $\gamma : \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathrm{diag}^{-1} \mathcal{O}_{X\times X}$, of sheaves on X, such that the following commutes:

$$\begin{tikzcd} \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[rr, "\gamma"] \arrow[dr, "\pi"'] & & \mathrm{diag}^{-1} \mathcal{O}_{X\times X} \arrow[dl] \\ & \mathcal{O}_X & \end{tikzcd}$$

We claim that γ is injective. If $U \subseteq X$ is an open set of *smooth* points, then $\alpha_U : \mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \to \mathcal{O}_{X\times X}(U \times U)$ is injective. In general, as X is reduced,

there is a dense open set $V \subseteq U$, such that V is a set of smooth points of X. We have the following commuting diagram

$$\begin{tikzcd}
\mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \arrow[r, "\alpha_U"] \arrow[d] & \mathcal{O}_{X\times X}(U \times U) \arrow[d] \\
\mathcal{O}_X(V) \otimes_{\mathbb{C}} \mathcal{O}_X(V) \arrow[r, "\alpha_V"] & \mathcal{O}_{X\times X}(V \times V)
\end{tikzcd}$$

where the vertical arrows are the restriction maps. As V is dense in U, these vertical arrows are injective, and α_V is injective by the smoothness of V. Hence for each open set $U \subseteq X$, α_U is injective. Taking limits we have that for all $p \in X$, $\alpha_p : (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X)_p \to \mathcal{O}_{X\times X,(p,p)}$ is injective. On the other hand,

$$\mathcal{O}_{X\times X,(p,p)} = \lim_{p\in U} \mathcal{O}_{X\times X}(U\times U) = \lim_{p\in U} \lim_{V\supseteq \mathrm{diag}(U)} \mathcal{O}_{X\times X}(V) = (\mathrm{diag}^{-1}\mathcal{O}_{X\times X})_p$$

and so $\beta_p : \mathcal{O}_{X\times X,(p,p)} \to (\mathrm{diag}^{-1}\mathcal{O}_{X\times X})_p$ is an isomorphism. It follows that $\gamma_p : (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X)_p \to (\mathrm{diag}^{-1}\mathcal{O}_{X\times X})_p$ is injective, as claimed.

Note that for $p \in X$ a smooth point, γ_p is given by the canonical isomorphism $\mathcal{O}_{X,p} \otimes_{\mathbb{C}} \mathcal{O}_{X,p} \xrightarrow{\approx} \mathcal{O}_{X\times X,(p,p)}$. Hence if X is smooth, γ is an isomorphism.

In any case, we have the following commuting diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathrm{diag}^{-1}\mathcal{O}_{X\times X} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r] & \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[u, "\gamma"] \arrow[ur, "\pi"'] & &
\end{tikzcd}$$

It follows that under the injection γ, $\mathcal{J} = \mathcal{I} \cap (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X)$. Hence $\mathcal{J}^{n+1} \subseteq \mathcal{I}^{n+1} \cap (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X)$. It follows that γ induces a (not necessarily injective) map

$$Jet_X^{(n)} = (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X)/\mathcal{J}^{n+1} \longrightarrow \mathrm{diag}^{-1}\mathcal{O}_{X\times X}/\mathcal{I}^{n+1} = \mathcal{P}_X^{(n)}$$

which is an isomorphism when X is smooth.

That this is $\mathcal{O}_X$-linear follows from the following commuting diagram

$$\begin{tikzcd}
\mathcal{O}_{X,p} \otimes_{\mathbb{C}} \mathcal{O}_{X,p} \arrow[rr, "\gamma_p"'] & & \mathcal{O}_{X\times X,(p,p)} \\
& \mathcal{O}_{X,p} \arrow[ul, "(p_1)_p"] \arrow[ur, "(\mathrm{pr}_1^*)_p"'] &
\end{tikzcd}$$

for each $p \in X$, which can be easily checked.

Similarly, the above diagram with p_1 and pr_1^* replaced by p_2 and pr_2^* also can be seen to commute. It follows that

$$\begin{tikzcd}
Jet_X^{(n)} \arrow[rr] & & \mathcal{P}_X^{(n)} \\
& \mathcal{O}_X \arrow[ul, "d_X^{(n)}"] \arrow[ur, "\mathrm{d}_X^{(n)}"'] &
\end{tikzcd}$$

commutes, as desired. □

Even when X is not a manifold, the above map is close to being an isomorphism between the sheaf of jets and the sheaf of principal parts. Namely, as we will see in the next section (Corollary 5.9), it induces an isomorphism on the "dual" sheaves.

(I use quotations as these sheaves are not necessarily locally free in the non-smooth case.) For now however, we can show that this induced map is at least an injection.

Corollary 4.3. *Suppose $(X, \mathcal{O}_X)$ is a reduced complex analytic space. Composition with the map $Jet_X^{(n)} \to \mathcal{P}_X^{(n)}$ given by Proposition 4.2 induces an injective map*

$$Hom_{\mathcal{O}_X}(\mathcal{P}_X^{(n)}, \mathcal{O}_X) \to Hom_{\mathcal{O}_X}(Jet_X^{(n)}, \mathcal{O}_X)$$

Proof. This is a corollary of the proof of Proposition 4.2, whose notation we continue to use. As $\mathcal{P}_X^{(n)}|_U = \mathcal{P}_U^{(n)}$, $Jet_X^{(n)}|_U = Jet_U^{(n)}$, and $\mathcal{O}_X|_U = \mathcal{O}_U$ for all open $U \subseteq X$, it suffices to work with global sections. Namely we show that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_X^{(n)}, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(Jet_X^{(n)}, \mathcal{O}_X)$$

is injective.

First of all note that $Jet_X^{(n)} \to \mathcal{P}_X^{(n)}$ was induced by $\gamma : \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathrm{diag}^{-1} \mathcal{O}_{X\times X}$ under the surjections $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \to Jet_X^{(n)}$ and $\mathrm{diag}^{-1} \mathcal{O}_{X\times X} \to \mathcal{P}_X^{(n)}$. It suffices therefore to show that composition with γ induces an injection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathrm{diag}^{-1} \mathcal{O}_{X\times X}, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X, \mathcal{O}_X)$$

Let $f : \mathrm{diag}^{-1} \mathcal{O}_{X\times X} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-linear map (of sheaves on X) such that $f \circ \gamma = 0$. We wish to show that $f = 0$. Note that for any $x \in X$,

$$(\mathrm{diag}^{-1} \mathcal{O}_{X\times X})_x = \lim_{x\in U} \mathcal{O}_{X\times X}(U \times U).$$

It suffices therefore to show that for all open $U \subseteq X$,

$$f_1(U) : \mathcal{O}_{X\times X}(U \times U) \to \mathrm{diag}^{-1} \mathcal{O}_{X\times X}(U) \xrightarrow{f(U)} \mathcal{O}_X(U)$$

is the zero map. Recall that for every smooth point $x \in U$, γ_x is an ismorphism, and hence f_x is the zero map. It follows that if U is smooth then $f_1(U) = 0$. In general, let $V \subseteq U$ be the dense open set of smooth points. We have

$$\begin{tikzcd} \mathcal{O}_{X\times X}(U \times U) \arrow[r, "f_1(U)"] \arrow[d] & \mathcal{O}_X(U) \arrow[d] \\ \mathcal{O}_{X\times X}(V \times V) \arrow[r, "f_1(V)"] & \mathcal{O}_X(V) \end{tikzcd}$$

where the vertical arrows are injections as V is dense in U. Since $f_1(V)$ is the zero map, so is $f_1(U)$, as desired. □

5. Differential operators

The constructions described here can be found in Kantor [5].

5.1. **Differential operators in an abstract setting.** Suppose $f : A \to B$ is a homomorphism of rings (commutative and unitary), and M, N are B-modules. We recursively define a class of A-linear maps from M to N, called *differential operators of order* $\leq n$ and denoted by $\mathrm{Diff}_{B/A}^{(n)}(M, N)$:

- $D \in \mathrm{Diff}_{B/A}^{(0)}(M, N)$ if D is B-linear.
- $D \in \mathrm{Diff}_{B/A}^{(n+1)}(M, N)$ if for every $b \in B$, $[D, b] \in \mathrm{Diff}_{B/A}^{(n)}(M, N)$, where $[D, b] : M \to N$ is defined by $[D, b](x) := D(bx) - bD(x)$.

We view $\mathrm{Diff}^{(n)}_{B/A}(M,N)$ as a B-module in the natural way: $(bD)(x) := bD(x)$.

Lemma 5.1. *An A-linear map $D : M \to N$ is a differential operator of order $\leq n$ if and only if for every $b_0, \dots, b_n \in B$, $[[\dots[[D,b_0],b_1],\dots],b_k]$ is the zero map.*

Proof. A straightforward induction on $n \geq 0$. $\square$

Let $\pi : B \otimes_A B \to B$ be the map given by $x \otimes y \mapsto xy$, and let $J := \ker \pi$. View $(B \otimes_A B)/J^{n+1}$ as a B-algebra via the map $B \xrightarrow{p_1} B \otimes_A B \to (B \otimes_A B)/J^{n+1}$. We also set $d : B \to (B \otimes_A B)/J^{n+1}$ to be the map $B \xrightarrow{p_2} B \otimes_A B \to (B \otimes_A B)/J^{n+1}$, which is *not* B-linear. Then $(B \otimes_A B)/J^{n+1}$ represents the functor $\mathrm{Diff}^{(n)}_{B/A}$:

Lemma 5.2. *For every B-module N, composition with d induces a B-linear isomorphism $\mathrm{Hom}_B((B \otimes_A B)/J^{n+1}, N) \xrightarrow{\approx} \mathrm{Diff}^{(n)}_{B/A}(B,N)$*

Proof. We will use $\overline{z}$ to denote the image of $z \in B \otimes_A B$ under the quotient map $B \otimes_A B \to (B \otimes_A B)/J^{n+1}$.

Let $h \in \mathrm{Hom}_B((B \otimes_A B)/J^{n+1}, N)$, and set $D := h \circ d : B \to N$. Note that for any $b, x \in B$,

$$\begin{aligned}
[D,b](x) &= D(bx) - bD(x) \\
&= h(\overline{1 \otimes bx}) - bh(\overline{1 \otimes x}) \\
&= h(\overline{1 \otimes bx} - \overline{b \otimes x}) \\
&= h((\overline{1 \otimes b} - \overline{b \otimes 1}) \cdot (\overline{1 \otimes x}))
\end{aligned}$$

Iterating this, we see that for any $b_0, \dots, b_n \in B$,

$$\begin{aligned}
[[\cdots[[D,b_0],b_1],\dots],b_n](x) &= h((\overline{1 \otimes b_0} - \overline{b_0 \otimes 1}) \cdots (\overline{1 \otimes b_n} - \overline{b_n \otimes 1}) \cdot (\overline{1 \otimes x})) \\
&= h(0 \cdot (\overline{1 \otimes x})) = 0
\end{aligned}$$

for all $x \in B$. Indeed, just note that each $(1 \otimes b_i - b_i \otimes 1) \in J$. By Lemma 5.1, we have that $D \in \mathrm{Diff}^{(n)}_{B/A}(B,N)$.

For injectivity, suppose $h_1, h_2 \in \mathrm{Hom}_B((B \otimes_A B)/J^{n+1}, N)$ and set $D_1 := h_1 \circ d$ and $D_2 := h_2 \circ d$. If $D_1 = D_2$, then for all $b \in B$, $h_1(\overline{1 \otimes b}) = h_2(\overline{1 \otimes b})$. Also as h_1, h_2 are B-linear, we have that

$$h_1(\overline{x \otimes y}) = h_1((\overline{x \otimes 1}) \cdot (\overline{1 \otimes y})) = xh_1(\overline{1 \otimes y}) = xh_2(\overline{1 \otimes y}) = h_2(\overline{x \otimes y})$$

for all $x, y \in B$. Since $(B \otimes_A B)/J^{n+1}$ is generated over A by elements of the form $\overline{x \otimes y}$, it follows that $h_1 = h_2$, as desired.

For surjectivity, let $D \in \mathrm{Diff}^{(n)}_{B/a}(B,N)$. Set $D_1 : B \otimes_A B \to N$ to be the B-linear map given by $x \otimes y \mapsto xD(y)$. Note that $D_1(xx' \otimes y) = xD_1(x' \otimes y)$ for all $x, x', y \in B$. Also, for any $x, y, b \in B$,

$$\begin{aligned}
D_1((1 \otimes b - b \otimes 1)(x \otimes y)) &= D_1(x \otimes by) - D_1(bx \otimes y) \\
&= xD(by) - bxD(y) \\
&= x[D,b](y)
\end{aligned}$$

It follows by iteration that for any $b_0, \dots, b_n \in B$

$$D_1((1 \otimes b_0 - b_0 \otimes 1) \cdots (1 \otimes b_n - b_n \otimes 1)(x \otimes y)) = x[[\dots[[D,b_0],b_1],\dots],b_n](y)$$

for all $x, y \in B$. By Lemma 5.1, and the fact that D is a differential operator of order $\leq n$, we see that D_1 vanishes on the ideal of $B \otimes_A B$ generated by all

elements of the form $(1\otimes b_0 - b_0\otimes 1)\cdots(1\otimes b_n - b_n\otimes 1)$. This ideal is exactly J^{n+1}. Hence D_1 induces a B-linear map $h:(B\otimes_{\mathbb{C}} B)/J^{n+1}\to B$. On the other hand, as $D_1(1\otimes x)=D(x)$ for all $x\in B$, we have that $D=h\circ d$.

This bijection $\mathrm{Hom}_B((B\otimes_A B)/J^{n+1},N)\to \mathrm{Diff}^{(n)}_{B/A}(B,N)$ is clearly additive. Moreover, for all $b,x\in B$, we have $bh\circ d(x)=bh(\overline{1\otimes x})=b(h\circ d)(x)$. Hence the bijection is a B-linear isomorphism, as desired. $\square$

Example 5.3. Consider $D\in \mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathbb{C}[z])$, where $z=(z_1,\dots,z_k)$ is a tuple of indeterminates. Then D can can be written uniquely as $D:=\displaystyle\sum_{|\alpha|\le n} a_\alpha D^\alpha$, where $\alpha\in\mathbb{N}^k$, $a_\alpha\in\mathbb{C}[z]$ and $D^\alpha:=\dfrac{1}{\alpha!}\displaystyle\prod_{1\le i\le k}(\frac{\partial}{\partial z_i})^{\alpha_i}$.

Proof. Indeed, $\mathbb{C}[z]\otimes_{\mathbb{C}}\mathbb{C}[z]=\mathbb{C}[z,z']$, and under this identification, the kernel of $\mathbb{C}[z]\otimes_{\mathbb{C}}\mathbb{C}[z]\to\mathbb{C}[z]$, J, is the one generated by elements of the form $z_i'-z_i$. Letting $u:=(z'-z)\bmod J^{n+1}$, we have that $\mathbb{C}[z,z']/J^{n+1}$ is freely generated over $\mathbb{C}[z]$ by elements of the form u^α, where $\alpha\in\mathbb{N}^k$ and $|\alpha|\le n$. Taking duals, we have that $\mathrm{Hom}_{\mathbb{C}[z]}(\mathbb{C}[z,z']/J^{n+1},\mathbb{C}[z])$ is freely generated over $\mathbb{C}[z]$ by $\{\tilde{D}^\alpha:|\alpha|\le n\}$, where $\tilde{D}^\alpha(\sum_\beta P_\beta(z)\cdot u_\beta)=P_\alpha(z)$. So by Lemma 5.2, $\mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathbb{C}[z])$ is freely generated over $\mathbb{C}[z]$ by $\{\tilde{D}^\alpha\circ d:|\alpha|\le n\}$, where $d:\mathbb{C}[z]\to\mathbb{C}[z,z']/J^{n+1}$ is given by $d(z)=z'\bmod J^{n+1}$. Fixing α we need to show that $\tilde{D}^\alpha\circ d=D^\alpha$ on $\mathbb{C}[z]$. Let $\beta\in\mathbb{N}^k$. Then $d(z^\beta)=(z+u)^\beta$. The u^α coefficient of this expression is $\binom{\beta}{\alpha}z^{\beta-\alpha}$. Hence $\tilde{D}^\alpha(d(z^\beta))=\binom{\beta}{\alpha}z^{\beta-\alpha}$. That is, as operators on $\mathbb{C}[z]$, $\tilde{D}^\alpha\circ d=D^\alpha$. $\square$

Example 5.4. Consider $D\in\mathrm{Diff}^{(n)}_{\mathcal{O}/\mathbb{C}}(\mathcal{O},\mathcal{O})$, where $\mathcal{O}$ is the ring of germs of holomorphic function at the origin in $\mathbb{C}^k$. Fix a system of co-ordinates, $z=(z_1,\dots,z_k)$. Then D can be written uniquely as $D:=\displaystyle\sum_{|\alpha|\le n}a_\alpha D^\alpha$, where $a_\alpha\in\mathcal{O}$ and the D^α are as in Example 5.3.

Proof. We first consider $\mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathcal{O})$. By Lemma 5.1 we have

$$\begin{aligned}\mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathcal{O}) &= \mathrm{Hom}_{\mathbb{C}[z]}((\mathbb{C}[z]\otimes_{\mathbb{C}}\mathbb{C}[z])/J^{n+1},\mathcal{O})\\ &= \mathrm{Hom}_{\mathbb{C}[z]}((\mathbb{C}[z]\otimes_{\mathbb{C}}\mathbb{C}[z])/J^{n+1},\mathbb{C}[z])\ \otimes_{\mathbb{C}[z]}\ \mathcal{O}\\ &= \mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathbb{C}[z])\ \otimes_{\mathbb{C}[z]}\ \mathcal{O}\end{aligned}$$

where the second equality uses that $(\mathbb{C}[z]\otimes_{\mathbb{C}}\mathbb{C}[z])/J^{n+1}$ is a free $\mathbb{C}[z]$-module of finite rank. It follows from Example 5.3 that $\mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathcal{O})$ is generated freely over $\mathcal{O}$ by the operators D^α. Hence the restriction map

$$\phi:\mathrm{Diff}^{(n)}_{\mathcal{O}/\mathbb{C}}(\mathcal{O},\mathcal{O})\to\mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathcal{O})$$

is surjective: every element of $\mathrm{Diff}^{(n)}_{\mathbb{C}[z]/\mathbb{C}}(\mathbb{C}[z],\mathcal{O})$ is of the form $\sum_{|\alpha|\le n}a_\alpha D^\alpha$, where $a_\alpha\in\mathcal{O}$, and hence is evidently the restriction of an operator from $\mathrm{Diff}^{(n)}_{\mathcal{O}/\mathbb{C}}(\mathcal{O},\mathcal{O})$. Finally, ϕ is injective since the restriction map from $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O},\mathcal{O})$ to $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[z],\mathcal{O})$ is injective by a theorem of Krull. $\square$

We can use the above example to understand, to some extent, differential operators on the germs of holomorphic functions on any complex analytic space:

Lemma 5.5. *Let $\mathcal{O}$ be the ring of germs of holomorphic functions at the origin in $\mathbb{C}^k$, and let $I \subseteq \mathcal{O}$ be an ideal that defines the germ of an analytic set $X_\circ$.*

(a) *Suppose $D \in \mathrm{Diff}^{(n)}_{\mathcal{O}/\mathbb{C}}(\mathcal{O}, \mathcal{O})$ is such that $D(I) \subseteq I$. Then the induced map $\overline{D} : \mathcal{O}_{X_\circ} \to \mathcal{O}_{X_\circ}$ is a differential operator of order $\leq n$.*

(b) *Conversely, if $E \in \mathrm{Diff}^{(n)}_{\mathcal{O}_{X_\circ}/\mathbb{C}}(\mathcal{O}_{X_\circ}, \mathcal{O}_{X_\circ})$, then there is $D \in \mathrm{Diff}^{(n)}_{\mathcal{O}/\mathbb{C}}(\mathcal{O}, \mathcal{O})$ such that $D(I) \subseteq I$ and $E = \overline{D}$.*

Proof. Let $r : \mathcal{O} \to \mathcal{O}/I = \mathcal{O}_{X_\circ}$ be the quotient map.

For part (a), as $D(I) \subseteq I$, it induces a $\mathbb{C}$-linear map $\overline{D} : \mathcal{O}_{X_\circ} \to \mathcal{O}_{X_\circ}$. That this map is in fact a differential operator of order $\leq n$ can be checked rather easily using the recursive definition of $\mathrm{Diff}^{(n)}_{\mathcal{O}_{X_\circ}/\mathbb{C}}(\mathcal{O}_{X_\circ}, \mathcal{O}_{X_\circ})$.

We will prove part (b) by a series of claims.[3] Fix a local co-ordinate system $z := (z_1, \dots, z_k)$ for $\mathcal{O}$. Let $E \in \mathrm{Diff}^{(n)}_{\mathcal{O}_{X_\circ}/\mathbb{C}}(\mathcal{O}_{X_\circ}, \mathcal{O}_{X_\circ})$ and $D \in \mathrm{Diff}^{(n)}_{\mathcal{O}/\mathbb{C}}(\mathcal{O}, \mathcal{O})$.

Claim 1: If $r(D(z^\alpha)) = E(r(z^\alpha))$ for all $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq n$, then $r(D(f)) = E(r(f))$ for all polynomials $f \in \mathcal{O}$. The proof of this claim goes by induction on $n \geq 0$, the case of $n = 0$ being clear. Fixing $1 \leq i \leq k$, note that for all $\beta \in \mathbb{N}^k$ with $|\beta| \leq n-1$,

$$\begin{aligned} r([D, z_i](z^\beta)) &= r(D(z_i z^\beta)) - r(z_i D(z^\beta)) \\ &= E(r(z_i z^\beta)) - r(z_i)E(r(z^\beta)) \\ &= [E, r(z_i)](r(z^\beta)) \end{aligned}$$

Hence, by induction, $r([D, z_i](f)) = [E, r(z_i)](r(f))$ for all polynomials $f \in \mathcal{O}$. Now suppose $r(D(g)) = E(r(g))$ for *some* polynomial $g \in \mathcal{O}$. Then,

$$\begin{aligned} r(D(z_i g)) &= r(z_i D(g) + [D, z_i](g)) \\ &= r(z_i)E(r(g)) + [E, r(z_i)](r(g)) \\ &= E(r(z_i g)) \end{aligned}$$

This holds for all $i = 1, \dots, k$. By iterating, we conclude that $r(D(f)) = E(r(f))$ for all polynomials $f \in \mathcal{O}$, proving Claim 1.

Claim 2: If $r(D(f)) = E(r(f))$ for all polynomials $f \in \mathcal{O}$, then $r \circ D = E \circ r$. Let $g \in \mathcal{O}$. For each $m > 0$, we may write $g = f_m + h_m$ where f_m is a polynomial and $h_m \in \mathcal{M}^m$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$. Hence,

$$r(D(g)) - E(r(g)) = r(D(h_m)) - E(r(h_m)).$$

Now, it is not hard to see that as $h_m \in \mathcal{M}^m$, $D(h_m) \in \mathcal{M}^{m-n}$ – where we set $\mathcal{M}^\ell = 0$ for $\ell < 0$. So, $r(D(h_m)) \in \mathcal{M}^{m-n}_{X_\circ}$, where $\mathcal{M}_{X_\circ}$ is the maximal ideal of $\mathcal{O}_{X_\circ}$. Similarly, $E(r(h_m)) \in \mathcal{M}^{m-n}_{X_\circ}$, and so $r(D(g)) - E(r(g)) \in \mathcal{M}^{m-n}_{X_\circ}$ for all $m > 0$. As $\bigcap_{m>0} \mathcal{M}^{m-n}_{X_\circ} = 0$, we have shown that $r(D(g)) = E(r(g))$ for all $g \in \mathcal{O}$, proving Claim 2.

Combining Claims 1 and 2, it suffices, in order to prove part (b), to find $D \in \mathrm{Diff}_{\mathcal{O}/\mathbb{C}}(\mathcal{O}, \mathcal{O})$ such that $r(D(z^\alpha)) = E(r(z^\alpha))$ for all $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq n$. For

[3] As the argument in Kantor seems circular, we give a different proof.

each such α, let $a_\alpha \in \mathcal{O}$ be such that $E(r(z^\alpha)) = r(a_\alpha)$. Now set $D := \sum_{|\alpha|\le k} a_\alpha D^\alpha$, with notation as in Example 5.4. This D works. $\square$

5.2. **Sheaves of differential operators.** The following construction is used by Campana in [1] toward the same ends as Fujiki uses principal parts in [2].

Definition 5.6. Suppose X is a complex analytic space. The presheaf on X given by $U \mapsto \mathrm{Diff}^{(n)}_{\mathcal{O}_X(U)/\mathbb{C}}(\mathcal{O}_X(U), \mathcal{O}_X(U))$ is in fact a sheaf of $\mathcal{O}_X$-modules; it is a subsheaf of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$. We denote this sheaf by $\mathcal{D}iff^{(n)}_X$ and call it the *sheaf of differential operators on X of order $\le n$.*

Proposition 5.7. *Suppose X is a complex analytic space. Composition with the map $d^{(n)}_X : \mathcal{O}_X \to \mathcal{J}et^{(n)}_X$ from Remark 4.1 induces an $\mathcal{O}_X$-linear isomorphism of sheaves on X,*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}et^{(n)}_X, \mathcal{O}_X) \xrightarrow{\approx} \mathcal{D}iff^{(n)}_X.$$

Proof. Fix an open set $U \subseteq X$. Let $\mathcal{J}(U) := \ker[\mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \to \mathcal{O}_X(U)]$ where the map is the one given by $f \otimes g \mapsto fg$. By Lemma 5.2 (with $A := \mathbb{C}$ and $B := N := \mathcal{O}_X(U)$) we know that composition with

$$\mathcal{O}_X(U) \xrightarrow{p_2(U)} \mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U)/\mathcal{J}(U)^{n+1}$$

induces an $\mathcal{O}_X(U)$-linear isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U)\otimes_{\mathbb{C}}\mathcal{O}_X(U)/\mathcal{J}(U)^{n+1}, \mathcal{O}_X(U)) \xrightarrow{\approx} \mathrm{Diff}^{(n)}_{\mathcal{O}_X(U)/\mathbb{C}}(\mathcal{O}_X(U), \mathcal{O}_X(U)).$$

This determines a presheaf isomorphism $s : \mathcal{H} \xrightarrow{\approx} \mathcal{D}iff^{(n)}_X$, where $\mathcal{H}$ is the presheaf $U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U)/\mathcal{J}(U)^{n+1}, \mathcal{O}_X(U))$. In particular, as $\mathcal{D}iff^{(n)}_X$ is a sheaf, so is $\mathcal{H}$. Now there is a canonical map $\alpha : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}et^{(n)}_X, \mathcal{O}_X) \to \mathcal{H}$ which is an isomorphism at stalks. As $\mathcal{H}$ is a sheaf, α is an isomorphism. Hence $s \circ \alpha : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}et^{(n)}_X, \mathcal{O}_X) \to \mathcal{D}iff^{(n)}_X$ is an isomorphism. By construction, it is the map induced by composing with $d^{(n)}_X : \mathcal{O}_X \to \mathcal{J}et^{(n)}_X$. $\square$

Proposition 5.8. *Suppose X is a reduced complex analytic space. Composition with the map $\mathrm{d}^{(n)}_X : \mathcal{O}_X \to \mathcal{P}^{(n)}_X$ from Remark 3.1 induces an $\mathcal{O}_X$-linear isomorphism of sheaves on X,*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{(n)}_X, \mathcal{O}_X) \xrightarrow{\approx} \mathcal{D}iff^{(n)}_X.$$

Proof. Let $\theta : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{(n)}_X, \mathcal{O}_X) \to \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ be the map induced by composition with $\mathrm{d}^{(n)}_X$. By Proposition 4.2, this factors as

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{(n)}_X, \mathcal{O}_X) \arrow[rr] \arrow[dr, "\theta"'] & & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}et^{(n)}_X, \mathcal{O}_X) \arrow[dl] \\
& \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X) &
\end{tikzcd}$$

where $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}et^{(n)}_X, \mathcal{O}_X) \to \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ is composition with $d^{(n)}_X$. Moreover, by Corollary 4.3, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{(n)}_X, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}et^{(n)}_X, \mathcal{O}_X)$ is injective. It follows from Proposition 5.7, that θ is an injection of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{(n)}_X, \mathcal{O}_X)$ in $\mathcal{D}iff^{(n)}_X$.

Fixing $x \in X$, it suffices to show the surjectivity of θ onto $\mathit{Diff}_X^{(n)}$, locally at x. Let $\tilde{X}$ be a sufficiently small open neighbourhood of x, such that $\tilde{X}$ is (isomorphic to) an analytic set in some open disc $U \subseteq \mathbb{C}^k$. We may assume x corresponds to the origin in $\mathbb{C}^k$. Let $E \in \mathit{Diff}_{X,x}^{(n)}(\tilde{X})$. Taking $\tilde{X}$ and U sufficiently small, we know by Lemma 5.5, that E is induced by some $D \in \mathrm{Diff}^{(n)}_{\mathcal{O}_k(U)/\mathbb{C}}(\mathcal{O}_k(U), \mathcal{O}_k(U))$, where $\mathcal{O}_k := \mathcal{O}_{\mathbb{C}^k}$ is the sheaf of germs of holomorphic functions on $\mathbb{C}^k$. By smoothness of U, we know that D factors through the principal parts

$$\begin{tikzcd}
\mathcal{O}_k(U) \arrow[rr, "D"] \arrow[dr, "\mathrm{d}_U^{(n)}"'] & & \mathcal{O}_k(U) \\
& \mathcal{P}^{(n)}_{\mathbb{C}^k}(U) \arrow[ur, "D_1"'] &
\end{tikzcd}$$

where $D_1 \in \mathrm{Hom}_{\mathcal{O}_k(U)}(\mathcal{P}^{(n)}_{\mathbb{C}^k}(U), \mathcal{O}_k(U))$. (Indeed, $\mathcal{P}^{(n)}_{\mathbb{C}^k}(U) = \mathit{Jet}^{(n)}_{\mathbb{C}^k}(U)$ and apply Proposition 5.7.) Let I be the kernel of the natural surjection $\mathcal{O}_k(U) \to \mathcal{O}_X(\tilde{X})$ – that is, I is the defining ideal of the analytic set $\tilde{X}$ in U. Note that the kernel of the induced surjection $\mathcal{P}^{(n)}_{\mathbb{C}^k}(U) \to \mathcal{P}^{(n)}_X(\tilde{X})$ is generated by $\mathrm{d}_U^{(n)}(I)$ and $\mathrm{pr}_1^*(I)$. Since D_1 is $\mathcal{O}_k(U)$-linear, it takes $\mathrm{pr}_1^*(I)$ back to I. On the other hand as $D(I) \subseteq I$, D_1 also takes $\mathrm{d}_U^{(n)}(I)$ to I. It follows that D_1 induces an $\mathcal{O}_X(\tilde{X})$-linear map $E_1 : \mathcal{P}^{(n)}_X(\tilde{X}) \to \mathcal{O}_X(\tilde{X})$, and we have:

$$\begin{tikzcd}
\mathcal{O}_k(U) \arrow[rr, "D"] \arrow[dr, "\mathrm{d}_U^{(n)}"'] \arrow[dddd] & & \mathcal{O}_k(U) \arrow[dddd] \\
& \mathcal{P}^{(n)}_{\mathbb{C}^k}(U) \arrow[ur, "D_1"'] \arrow[dd] & \\
& & \\
& \mathcal{P}^{(n)}_X(\tilde{X}) \arrow[dr, "E_1"] & \\
\mathcal{O}_X(\tilde{X}) \arrow[rr, "E"'] \arrow[ur, "\mathrm{d}_{\tilde{X}}^{(n)}"] & & \mathcal{O}_X(\tilde{X})
\end{tikzcd}$$

That is, $\theta_{\tilde{X}}$ takes E_1 to E, and we have shown that θ is locally surjective (and hence surjective) onto $\mathit{Diff}_X^{(n)}$. □

Corollary 5.9. *Suppose $(X, \mathcal{O}_X)$ is a reduced complex analytic space. Composition with the map $\mathit{Jet}_X^{(n)} \to \mathcal{P}_X^{(n)}$ given by Proposition 4.2 induces an isomorphism*

$$\mathit{Hom}_{\mathcal{O}_X}(\mathcal{P}_X^{(n)}, \mathcal{O}_X) \xrightarrow{\approx} \mathit{Hom}_{\mathcal{O}_X}(\mathit{Jet}_X^{(n)}, \mathcal{O}_X)$$

Proof. This follows from Propositions 5.7 and 5.8. □

Remark 5.10. Corollary 5.9 does *not* imply that $\mathit{Jet}_X^{(n)} \approx \mathcal{P}_X^{(n)}$. If these sheaves were locally free then it would – by taking duals. In the case that X is smooth the sheaves of principal parts and of jets are locally free, but in that case we already know $\mathit{Jet}_X^{(n)} \approx \mathcal{P}_X^{(n)}$ by Proposition 4.2.

6. Applications: the Dichotomoy Theorem

In this section we describe how the sheaves of principal parts and differential operators are used by Fujiki [2] and Campana[1], respectively, to prove a theorem which Pillay [7] has observed implies the dichotomy theorem for minimal types in the theory of compact complex analytic spaces.

Definition 6.1. A morphism $f : X \to S$ of (possible non-reduced) irreducible complex analytic spaces is *projective* if it factors through a closed embedding into a projective linear space over S. That is, there exists a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathbb{P}(\mathcal{F}) \arrow[dl] \\ & S & \end{tikzcd}$$

where h is a closed embedding and $\mathcal{F}$ is a coherent analytic sheaf on S. We say that $g : Y \to S$ is *Moishezon* if there is a commutative diagram:

$$\begin{tikzcd} Y \arrow[rr, "\alpha"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

where α is a bimeromorphism and $f : X \to Y$ is projective.

A discussion of the model-theoretic significance of Moishezon morphisms, namely its relationship to *internality in* $\mathbb{P}$, can be found in [6].

We work in the following context: Suppose X and S are reduced and irreducible compact complex analytic spaces, and $Z \subseteq S \times X$ is an irreducible analytic subset. Let $\rho : Z \to S$ be the first co-ordinate projection map. We view:

$$\begin{tikzcd} Z \subseteq S \times X \arrow[d, "\rho"] \\ S \end{tikzcd}$$

as a family of analytic subspaces of X parametrised by S: namely the family of fibres $Z_s := \{x \in X : (s, x) \in Z\}$ for $s \in S$. *We make the following standing assumptions*:

(1) The map ρ is surjective and there is a dense Zariski open set $W \subseteq S$ such that Z_w is reduced and irreducible for all $w \in W$.
(2) $\rho : Z \to S$ is a *canonical family*: There is a dense Zariski open set $V \subseteq S$ such that if $v, v' \in V$ and $Z_v = Z_{v'}$, then $v = v'$.

In the language of types, these assumptions correspond to saying that we are studying a stationary type over its canonical base.

We aim to expose the following result of Campana [1]/Fujiki [2]:

Theorem 6.2. *The second co-ordinate projection $\pi : Z \to X$ is Moishezon.*

6.1. **Fujiki's approach.** Fix $n \geq 0$ and $a \in X$. Let $\Delta_X^{(n)} \to X$ be the map $\mathrm{pr}_1 \circ \mathrm{diag}^{(n)}$. Proposition 3.3 tells us that the (sheaf-theoretic) fibre of $\Delta_X^{(n)} \to X$ over a, which we will denote by $\Delta_{X,a}^{(n)}$, is the non-reduced space $(\{a\}, \mathcal{O}_{X,a}/\mathcal{M}_{x,a}^{n+1})$. That is, $\Delta_{X,a}^{(n)}$ is the nth infinitesimal neighbourhood of a in X.

We wish to view $\pi^{-1}(a) \subseteq Z$ as parametrising a family of analytic subsets of $\Delta_{X,a}^{(n)}$. The basic idea is as follows: for $c \in \pi^{-1}(a)$, $Z_{\rho(c)}$ is an analytic subset of X passing through a, and hence the nth infinitesimal neighbourhood of a in $Z_{\rho(c)}$ is a subspace of $\Delta_{X,a}^{(n)}$. We can view $\pi^{-1}(a)$ as parametrising the family of nth infinitesimal neighbourhood of a in $Z_{\rho(c)}$, for $c \in \pi^{-1}(a)$. We will do this formally, and uniformly in a, as follows.

Identify $Z \times_S Z$ with a subspace of $Z \times_X (X \times X)$ in the natural way:

$$Z \times_S Z \subseteq Z \times_S (S \times X) \approx Z \times X \approx Z \times_X (X \times X)$$

Also identify $\Delta_X^{(n)}$ with a subspace of $X \times X$ via $\mathrm{diag}^{(n)}$. Now define

$$Y_n := (Z \times_S Z) \cap (Z \times_X \Delta_X^{(n)})$$

where the intersection is taking place inside $Z \times_X (X \times X)$. We have the co-ordinate projection $Z \times_X (X \times X) \to Z$ which restricts to:

$$\begin{array}{c} Y_n \subseteq Z \times_X \Delta_X^{(n)} \\ \big\downarrow {\scriptstyle \gamma_n} \\ Z \end{array}$$

This makes $Y_n \to Z$ into a family of analytic subspaces of $\Delta_X^{(n)}$ over X. Explicitly, for each $a \in X$, restricting everything to a we obtain:

$$\begin{array}{c} (Y_n)_{\pi^{-1}(a)} \subseteq \pi^{-1}(a) \times \Delta_{X,a}^{(n)} \\ \big\downarrow {\scriptstyle \gamma_n} \\ \pi^{-1}(a) \end{array}$$

and for each $c \in \pi^{-1}(a) \cap W$ the (sheaf-theoretic) fibre $Y_{n,c}$ is the nth infinitesimal neighbourhood of a in $Z_{\rho(c)}$ viewed as a subspace of $\Delta_{X,a}^{(n)}$.

For some dense Zariski open set $U_n \subseteq Z$, $\gamma_n : Y_n \to Z$ is flat over U_n. So U_n is a parameter space for a flat family of analytic subspaces of $\Delta_X^{(n)}$ over X. But there is a *universal parameter space for flat families of analytic subspaces of* $\Delta_X^{(n)}$ *over* X – namely the relative Douady space of $\Delta_X^{(n)}$ over X. What this means is that there is a reduced complex analytic space over X, $\mathcal{D}(\Delta_X^{(n)}/X) \to X$, and a diagram of holomorphic maps:

$$\begin{tikzcd} U_n \arrow[rr, "\tau_n"] \arrow[dr, "\pi"'] & & \mathcal{D}(\Delta_X^{(n)}/X) \arrow[dl] \\ & X & \end{tikzcd}$$

such that for $c, c' \in U_n$, $\tau_n(c) = \tau_n(c')$ if and only if $Y_{n,c} = Y_{n,c'}$. Moreover, by Hironaka's Flattening Theorem, τ_n extends to a meromorphic map $Z \to \mathcal{D}(\Delta_X^{(n)}/X)$, which we will also (abusively) denote as τ_n.

A key fact is that every component C of $\mathcal{D}(\Delta_X^{(n)}/X)$ is compact and the restricted map $C \to X$ is projective. This is because $\Delta_X^{(n)} \to X$ is projective (indeed, it is finite), and is proved in Lemma 5 of [2]. Hence to show that π is Moishezon (i.e. to prove Theorem 6.2) it will suffice to show that for some $n > 0$, τ_n is a bimeromorphism with its image.

Remark 6.3. Note that $\gamma_n : Y_n \to Z$ does not live definably in the many-sorted structure $\mathcal{A}$ of compact complex analytic spaces. This is because Y_n is not reduced, and if we take its reduction then we lose all the information we require – namely, the infinitesimal neighbourhoods. However, the map $\tau_n : Z \to \mathcal{D}(\Delta_X^{(n)}/X)$ is definable and it captures the information given by $\gamma_n : Y_n \to Z$ – since for $c, c' \in U_n$, $\tau_n(c) = \tau_n(c')$ if and only if $Y_{n,c} = Y_{n,c'}$. This is somehow the point.

We now show that for some $n > 0$, τ_n is a bimeromorphism with its image.

First note what it means to prove that some τ_n is a bimeromorphism with its image. For $c, c' \in U_n$, supposing $\tau_n(c) = \tau_n(c')$ one asks whether $c = c'$. As $\pi(c) = \pi(c')$ (since τ_n is over X), this is the same as asking if $\rho(c) = \rho(c')$. But $\rho : Z \to S$ is a canonical family (standing assumption #2), so after replacing U_n with $U_n \cap V$, $\rho(c) = \rho(c')$ if and only if $Z_{\rho(c)} = Z_{\rho(c')}$. On the other hand, from the definition of τ_n we have that $\tau_n(c) = \tau_n(c')$ if and only if $Y_{n,c} = Y_{n,c'}$; and the latter means that the nth infinitesimal neighbourhood of $a := \pi(c) = \pi(c')$ in $Z_{\rho(c)}$ and in $Z_{\rho(c')}$ agree. So to say that τ_n is a bimeromorphism with its image is to say (essentially) that the analytic subsets of X from the family $Z \to S$ are distinguishable by the nth infinitesimal neighbourhoods of a common point.

Claim 6.4. *For irreducible analytic sets $A, B \subseteq X$ with $A \cap B \neq \emptyset$, $A = B$ if and only if at some common point the nth infinitesimal neighbourhoods agree for all n.*

Proof. Only the right to left direction requires proof. We will show that $A \subseteq B$ and concluded by symmetry that $A = B$. Let $a \in A \cap B$. As $A \cap B$ is an analytic subset of A, and as A is irreducible, it suffices to show that in some non-empty subset of X containing a, say U, $(A \cap U) \subseteq (B \cap U)$. This latter condition is local and so we may assume that A and B are analytic subsets of the unit disc in some $\mathbb{C}^k$, and that a is the origin. Suppose $f \in \mathcal{O}$ is a germ of a holomorphic function at the origin which vanishes on the germ of B at the origin. Hence the image of f in $\mathcal{O}_B$ – the ring of germs of holomorphic functions on B at the origin – is contained in $\mathcal{M}_B^{n+1}$ for all $n+1$. As the nth infinitesimal neighbourhoods of the origin in A and B agree for all n, this implies that the image of f in $\mathcal{O}_A$ is contained in $\mathcal{M}_A^{n+1}$ for all $n+1$. As $\bigcap_n \mathcal{M}_A^{n+1} = 0$, we have that the image of f in $\mathcal{O}_A$ is zero. It follows that the defining ideal of the germ of B is contained in that of the germ of A. Hence, for some open set U about the origin, $(A \cap U) \subseteq (B \cap U)$, as desired. □

Claim 6.5. *Let $m \leq n$ and $c, c' \in U_n \cap U_m$. If $\tau_n(c) = \tau_n(c')$ then $\tau_m(c) = \tau_m(c')$.*

Proof. Suppose $\tau_n(c) = \tau_n(c')$ and let $a := \pi(c) = \pi(c')$. So the nth infinitesimal neighbourhoods of a in $Z_{\rho(c)}$ and $Z_{\rho(c')}$ agree. It is clear from the definitions that so then do the mth infinitesimal neighbourhoods. Hence $\tau_m(c) = \tau_m(c')$. □

Theorem 6.2 now follows by an elementary model-theoretic argument.

Let $U := \bigcap_n U_n$ and let $V, W \subseteq S$ be the dense Zariski open sets given by the standing assumptions 1 and 2. Let $\Phi(x, y)$ be the partial type which says that

$$x, y \in U \cap \rho^{-1}(V) \cap \rho^{-1}(W);\ \tau_n(x) = \tau_n(y) \text{ for all } n > 0; \text{ and } x \neq y.$$

If $(c, c') \models \Phi(x, y)$ then the nth infinitesimal neighbourhoods of $a := \pi(c) = \pi(c')$ in $Z_{\rho(c)}$ and $Z_{\rho(c')}$ agree for all n. These are irreducible analytic subsets of X (as $\rho(c), \rho(c') \in W$), and so $Z_{\rho(c)} = Z_{\rho(c')}$ (by Claim 6.4), and so $\rho(c) = \rho(c')$ (as $\rho(c), \rho(c') \in V$), and so $c = c'$. But this is a contradiction. Hence $\Phi(x, y)$ is not realised in Z. By ω_1-compactness (note that $\Phi(x, y)$ involves only countably many formulae) there is some $n > 0$ such that for all $c, c' \in U_1 \cap \cdots \cap U_n \cap \rho^{-1}(V) \cap \rho^{-1}(W)$, if $c \neq c'$ then $\bigvee_{i=1}^{n} \tau_i(c) \neq \tau_i(c')$. By Claim 6.5, $\tau_n(c) \neq \tau_n(c')$. So τ_n is a bimeromorphism with its image, as desired. $\square$

6.2. **Campana's approach.** We conclude with a brief sketch of the idea, as we understand it, behind Campana's proof of Theorem 6.2.

The techniques in [1] differ from Fujiki's in essentially two ways. First of all, the analytic *space* of infinitesimal neighbourhoods $\Delta_X^{(n)}$ over X is replaced by the *sheaf* of differential operators $\mathit{Diff}_X^{(n)}$ on X. By Proposition 5.8 this amounts to taking the dual to the structure sheaf of $\Delta_X^{(n)}$. Secondly, Douady spaces are replaced with Grassmanians.

The Grassmanian of a coherent analytic sheaf on X is a complex analytic space over X, introduced by Grothendieck in [3], which relativises the usual notion of Grassmanian for complex vector spaces. Fixing $n > 0$, the role of $\mathcal{D}(\Delta_X^{(n)}/X) \to X$ will now be played by $\mathrm{Grass}_{r_n}(\mathit{Diff}_X^{(n)}) \to X$ for some positive integer r_n (to be determined below). Fixing $a \in X$, the fibre of $\mathrm{Grass}_{r_n}(\mathit{Diff}_X^{(n)})$ over a can be identified with the Grassmanian of r_n-planes (that is, co-dimension r_n subspaces) of the complex vector space $\mathit{Diff}_{X,a}^{(n)} \otimes_{\mathcal{O}_{X,a}} \mathbb{C}$. For $a \in X$ smooth, the latter can be computed as follows:

$$\begin{aligned}
\mathit{Diff}_{X,a}^{(n)} \otimes_{\mathcal{O}_{X,a}} \mathbb{C} &= \mathit{Hom}_{\mathcal{O}_X}(\mathcal{P}_X^{(n)}, \mathcal{O}_X)_a \otimes_{\mathcal{O}_{X,a}} \mathbb{C} \\
&= \mathrm{Hom}_{\mathcal{O}_{X,a}}(\mathcal{P}_{X,a}^{(n)}, \mathcal{O}_{X,a}) \otimes_{\mathcal{O}_{X,a}} \mathbb{C} \\
&= \mathrm{Hom}_{\mathbb{C}}(\mathcal{P}_{X,a}^{(n)} \otimes_{\mathcal{O}_{X,a}} \mathbb{C}\ ,\ \mathbb{C}) \\
&= \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_{X,a}/\mathcal{M}_{X,a}^{n+1}\ ,\ \mathbb{C})
\end{aligned}$$

where the first equality is by Proposition 5.8, the second by the choice of $a \in X$ smooth, and the last by Proposition 3.3. Hence

$$\mathrm{Grass}_{r_n}(\mathit{Diff}_X^{(n)})_a = \mathrm{Grass}_{r_n}([\mathcal{O}_{X,a}/\mathcal{M}_{X,a}^{n+1}]^*)$$

where $[\cdots]^*$ denotes complex vector space duals.

Instead of a meromorphic map $\tau_n : Z \to \mathcal{D}(\Delta_X^{(n)}/X)$, we are now interested in producing a natural meromorphic map μ_n;

$$\begin{tikzcd} Z \arrow[rr, "\mu_n"] \arrow[dr, "\pi"'] & & \mathrm{Grass}_{r_n}(\mathit{Diff}_X^{(n)}) \arrow[dl] \\ & X & \end{tikzcd}$$

For $a \in X$ smooth, the idea is to view $\pi^{-1}(a)$ as parametrising a family of linear subspaces of $[\mathcal{O}_{X,a}/\mathcal{M}_{X,a}^{n+1}]^*$. Given $c \in \pi^{-1}(a)$, $Z_{\rho(c)}$ is an analytic set in X passing through a. Hence $[\mathcal{O}_{Z_{\rho(c)},a}/\mathcal{M}_{Z_{\rho(c)},a}^{n+1}]^*$ is a subspace of $[\mathcal{O}_{X,a}/\mathcal{M}_{X,a}^{n+1}]^*$. For a and c sufficiently general the codimension of this subspace is constant, say r_n. So $[\mathcal{O}_{Z_{\rho(c)},a}/\mathcal{M}_{Z_{\rho(c)},a}^{n+1}]^*$ corresponds to a point in $\mathrm{Grass}_{r_n}([\mathcal{O}_{X,a}/\mathcal{M}_{X,a}^{n+1}]^*) = \mathrm{Grass}_{r_n}(\mathit{Diff}_X^{(n)})_a$. The map μ_n will associate to c this point.

One must ensure that the partial map μ_n obtained in this way is indeed meromorphic. As we are unable to add anything enlightening on this point to what can be found in [1], we leave the reader to look there for the formal construction of μ_n.

The (dual to the) argument for Claim 6.4 shows that if $A, B \subset X$ are irreducible analytic sets with $A \cap B \neq \emptyset$, then $A = B$ if and only if for some $a \in A \cap B$ and all $n > 0$, $[\mathcal{O}_{A,a}/\mathcal{M}_{A,a}^{n+1}]^* = [\mathcal{O}_{B,a}/\mathcal{M}_{B,a}^{n+1}]^*$ as subspaces of $[\mathcal{O}_{X,a}/\mathcal{M}_{X,a}^{n+1}]^*$. Then, using ω-compactness as before, one shows that for sufficiently large n, μ_n is a bimeromorphism with its image. As $\mathrm{Grass}_{r_n}(\mathit{Diff}_X^{(n)}) \to X$ is projective (by the Plücker embedding, see [3]), this shows that $\pi : Z \to X$ is Moishezon, hence giving an alternative proof for Theorem 6.2. □

References

[1] F. Campana. Algébricité et compacité dans l'espace des cycles d'un espace analytique complexe. *Mathematische Annalen*, 251(1):7–18, 1980.

[2] A. Fujiki. On the Douady space of a compact complex space in the category $\mathcal{C}$. *Nagoya Mathematical Journal*, 85:189–211, 1982.

[3] A. Grothendieck. Techniques de construction en géométrie analytique V. Fibrés vectoriels, fibrés projectifs, fibrés en drapeaux. *Séminaire Henri Cartan*, 13(12), 1960/61.

[4] A. Grothendieck. Techniques de construction en géométrie analytique VII. Étude locale des morphisme: éléments de calcul infinitésimal. *Séminaire Henri Cartan*, 13(14), 1960/61.

[5] J.M. Kantor. Formes et opérateurs différentiels sur les espaces analytiques complexes. *Bull. Soc. Math. France Mém.*, 53:5–80, 1977.

[6] R. Moosa. A nonstandard Riemann existence theorem. *Transactions of the American Mathematical Society*, 356(5):1781–1797, 2004.

[7] A. Pillay. Model-theoretic consequences of a theorem of Campana and Fujiki. *Fundamenta Mathematicae*, 174(2):187–192, 2002.

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